Lecture 3

Equations solvable for x :-

If the solved form is $x = F(y,p)$, we differentiate w.r.to y.

We deduce the relation

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Solve this equation & obtain a relation of the form $\psi(y,p,c) = 0$.

Eliminate p between this last relation & the given equation.

The eliminate is the complete primitive.

The complete primitive may be given in the parametric form.

Example ① solve $p^2y + 2px = y$

Solution :- Solving for x

$$\therefore x = \frac{y}{2p}(1-p^2)$$

Differentiating w.r.to y,

$$\frac{1}{p} = \frac{1-p^2}{2p} + y\left\{-\frac{1}{2}\,\frac{1+p^2}{p^2}\cdot\frac{dp}{dy}\right\}$$

$$\Rightarrow \frac{1+p^2}{2p} = -\frac{1}{2}\,\frac{1+p}{p2}\,y\,\frac{dp}{dy}$$

$$\Rightarrow 1 = -\frac{y}{p}\,\frac{dp}{dy} \quad (\because 1+p^2 \neq 0)$$

$$\Rightarrow \frac{dy}{y} + \frac{dp}{p} = 0 \quad \therefore \; py = c$$

Eliminating p between the relation $p = \frac{c}{y}$ and the given equation $x = y(1-p^2)$, we obtain

$$y^2 = c^2 + 2cx$$

Ans.

Example ② solve $x = py - p^2$ —— ①

Solution:- Differentiating ① w.r.to y

$$\frac{1}{p} = p + y\frac{dp}{dy} - 2p\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} - p = (y - 2p)\frac{dp}{dy}$$

$$\Rightarrow \frac{dy}{dp}\left(\frac{1}{p} - p\right) = y - 2p$$

$$\Rightarrow \frac{dy}{dp} + \frac{y}{p - \frac{1}{p}} = \frac{2p}{p - \frac{1}{p}}$$

Which is a linear equation in y with p as its independent variable.

$$\therefore \text{I.F.} = e^{\int \frac{p}{p^2-1}dp} = e^{\frac{1}{2}\log(p^2-1)} = \sqrt{p^2-1}$$

Multiplying by the integrating factor & then integrating, we find

$$y\sqrt{p^2-1} = \int \frac{2p^2}{\sqrt{p^2-1}}\,dp + c$$

$$= 2\int \frac{p^2-1+1}{\sqrt{p^2-1}}\,dp + c$$

$$= 2\int \sqrt{p^2-1}\,dp + 2\int \frac{dp}{\sqrt{p^2-1}} + c$$

$$= p\sqrt{p^2-1} + \cosh^{-1}p + c$$

$$\therefore y = p + (c + \cosh^{-1}p)(p^2-1)^{-\frac{1}{2}}$$

This relation & the given equation

$$x = py - p^2$$

$$\therefore x = p(c + \cosh^{-1}p)(p^2-1)^{-\frac{1}{2}}$$

Ans.